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A monograph on the classification of the discrete subgroups of SU(4)

Amihay Hanany and Yang-Hui He*

Center for Theoretical Physics, Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, USA
E-mail: hanany@ctp.mit.edu, yhe@ctp.mit.edu

Abstract: We here present, in modern notation, the classification of the discrete finite subgroups of SU(4) as well as the character tables for the exceptional cases thereof (cf. http://pierre.mit.edu/~yhe/su4.ct). We hope this catalogue will be useful to works on string orbifold theories, quiver theories, WZW modular invariants, Gorenstein resolutions, nonlinear sigma-models as well as some recently proposed inter-connections among them.

Keywords: Conformal Field Models in String Theory, Brane Dynamics in Gauge Theories, Discrete and Finite Symmetries.

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Contents

1. Introduction

It is well known that the discrete finite subgroups of $\mathrm{SL}(n = 2,3;\mathbb{C})$ have been completely classified; works related to string orbifold theories and quiver theories have of late used these results (see for example [5, 6, 8, 13] as well as references therein). Conjectures regarding higher n have been raised and works toward finite subgroups of SU(4) are under way. Recent works by physicists and mathematicians alike further beckon for a classification of the groups, conveniently presented, in the case of SU(4) [14]. Compounded thereupon is the disparity of language under which the groups are discussed: the classification problem in the past decades has chiefly been of interest to either theoretical chemists or to pure mathematicians, the former of whom disguise them in Bravais crystallographic notation (e.g. [15]) while the latter abstract them in fields of finite characteristic (e.g. [16]). Subsequently, there is a need within the string theory community for a list of the finite subgroups of SU(4) tabulated in our standard nomenclature, complete with the generators and some brief but not overly-indulgent digression on their properties.

The motivations for this need are manifold. There has recently been a host of four dimensional finite gauge theories constructed by placing D3 branes on orbifold singularities [8]; brane setups have also been achieved for some of the groups [10]. In particular, a theory with $\mathcal{N} = 2, 1, 0$ supercharges respectively is obtained from a $\mathbb{C}^N/\{\Gamma \subset \mathrm{SU}(n = 2, 3, 4)\}$ singularity with $N = 2, 3$ (see [5] [8] and references therein). Now as mentioned above $n = 2, 3$ have been discussed, and $n = 4$ has yet to be fully attacked. This last case is of particular interest because it gives rise to an $\mathcal{N} = 0$, non-supersymmetric theory. On the one hand these orbifold theories provide interesting string backgrounds for checks on the AdS/CFT Correspondence [9]. On the other hand, toric descriptions for the abelian cases of the canonical Gorenstein singularities have been treated while the non-abelian still remain elusive [11]. Moreover, the quiver theories arising from these string orbifold theories (or equivalently, representation rings of finite subgroups of $\mathrm{SU}(n)$) have been hinted to be related to modular invariants of $\widehat{\mathrm{su}(n)}$-WZW models (or equivalently, affine characters of $\widehat{\mathrm{su}(n)}$) for arbitrary n [5, 7], and a generalised McKay Correspondence, which would also relate non-linear sigma models, has been suggested to provide a reason [6]. Therefore a need for the discrete subgroups of $\mathrm{SU}(4)$ arises in all these areas.

Indeed the work has been done by Blichfeldt [2] in 1917, or at least all the exceptional cases, though in an obviously outdated parlance and moreover with many infinite series being "left to the reader as an exercise". It is therefore the intent of the ensuing monograph to present the discrete subgroups Γ of $\mathrm{SL}(4, \mathbb{C})$ in a concise fashion, hoping it to be of use to impending work, particularly non-supersymmetric conformal gauge theories from branes on orbifolds, resolution of Gorenstein singularities in higher dimension, as well as $\widehat{\mathrm{su}(4)}$-WZW models.

Nomenclature Unless otherwise stated we shall adhere to the convention that Γ refers to a discrete subgroup of $\mathrm{SU}(n)$ (i.e., a finite collineation group), that $< x_1, \ldots, x_n >$ is a finite group generated by $\{x_1, \ldots, x_n\}$, that $H \triangleleft G$ means H is a normal subgroup of G, that S_n and A_n are respectively the symmetric and alternating permutation groups on n elements, and that placing $*$ next to a group signifies that it belongs to $\mathrm{SU}(4) \subset \mathrm{SL}(4; \mathbb{C})$.

2. Preliminary definitions

Let Γ be a finite discrete subgroup of the general linear group, i.e., $\Gamma \subset GL(n, \mathbb{C})$. From a mathematical perspective, quotient varieties of the form $\mathbb{C}^n/\Gamma$ may be constructed and by the theorem of Khinich and Watanabe [1, 3], the quotient is Gorenstein[1] if and only if Γ is in fact in $\mathrm{SL}(n, \mathbb{C})$. Therefore we would like to focus on the discrete subgroups of linear transformations *up to linear equivalence*, which are what has been dubbed in the old literature as *finite collineation groups* [2]. From a physics perspective, discrete subgroups of $\mathrm{SU}(n) \subset \mathrm{SL}(n; \mathbb{C})$ have been subject to

[1]That is, if there exists a nowhere-vanishing holomorphic n-form. These varieties thus provide local models of Calabi-Yau manifolds and are recently of great interest.

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investigation in the early days of particle phenomenology [4] and have lately been of renewed interest in string theory, especially in the context of orbifolds (see for example [5, 6, 8, 14]).

There are some standard categorisations of finite collineation groups [2, 3]. They first fall under the division of transitivity and intransitivity as follows:

Definition 1. *If the n variables upon which Γ acts as a linear transformation can be separated into 2 or more sets either directly or after a change of variables, such that the variables of each set are transformed into linear functions only of themselves, then Γ is called* intransitive*; it is called transitive otherwise.*

The transitive Γ can be further divided into the primitive and imprimitive cases:

Definition 2. *If for the transitive Γ the variables may be separated*[2] *into 2 or more sets such that the variables of each are transformed into linear functions of only those in any set according to the separation (either the same or different), then Γ is called* imprimitive*; it is called Primitive otherwise.*

Therefore in the matrix representation of the groups, we may naïvely construe intransitivity as being block-diagonalisable and imprimitivity as being block off-diagonalisable, whereby making primitive groups generically having no *a priori* zero entries. We give examples of an intransitive, a (transitive) imprimitive and a (transitive) primitive group, in their matrix forms, as follows:

$$\begin{array}{ccc}
\begin{pmatrix} \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix} &
\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \\
\text{Intransitive} & \text{Imprimitive} & \text{Primitive} \\
 & \multicolumn{2}{c}{\text{Transitive}}
\end{array}$$

Let us diagrammatically summarise all these inter-relations as is done in [3]:

$$\Gamma \begin{cases} \text{Intransitive} \\ \text{Transitive} \begin{cases} \text{Imprimitive} \\ \text{Primitive} \begin{cases} \text{Simple} \\ \text{Having normal primitive subgroups} \\ \text{Having normal intransitive subgroups} \\ \text{Having normal imprimitive subgroups} \end{cases} \end{cases} \end{cases}$$

In some sense the primitive groups are the fundamental building blocks and pose as the most difficult to be classified. It is those primitive groups that Blichfeldt presented, as linear transformations, in [2]. These groups are what we might call

[2] Again, either directly or after a change of variables.

exceptionals in the sense that they do not fall into infinite series, in analogy to the $E_{6,7,8}$ groups of SU(2). We present them as well as their sub-classifications first. Thereafter we shall list the imprimitive and intransitives, which give rise to a host of infinite series of groups, in analogy to the A_n and D_n of SU(2).

Let us take a final digression to clarify the so-called *Jordan notation*, which is the symbol ϕ commonly used in finite group theory. A linear group Γ often has its order denoted as $|\Gamma| = g\phi$ for positive integers g and ϕ; the ϕ signifies the order of the subgroup of homotheties, or those multiples of the identity which together form the center of the $\mathrm{SL}(n;\mathbb{C})$. We know that $\mathrm{SU}(n) \subset \mathrm{SL}(n;\mathbb{C})$, so a subgroup of the latter is not necessarily that of the former. In the case of $\mathrm{SL}(n=2,3;\mathbb{C})$, the situation is simple:[3] the finite subgroups belonged either to (A) $\mathrm{SU}(n=2,3)$, or to (B) the center-modded[4] $\mathrm{SU}(n=2,3)/\mathbb{Z}_{2,3}$, or (C) to both. Of course a group with order g in type (B) would have a natural lifting to type (A) and become a group of order g multiplied by $|\mathbb{Z}_2| = 2$ or $|\mathbb{Z}_3 = 3|$ respectively, which is now a finite subgroup of the full SU(2) or SU(3), implying that the Jordan ϕ is 2 or 3 respectively.

For the case at hand, the situation is slightly more complicated since 4 is not a prime. Therefore ϕ can be either 2 or 4 depending how one lifts with respect to the relation $\mathrm{SU}(4)/\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathrm{SO}(6)$ and we lose a good discriminant of whether or not Γ is in the full SU(4). To this end we have explicitly verified the unitarity condition for the group elements and will place a star $(*)$ next to those following groups which indeed are in the full SU(4). Moreover, from the viewpoint of string orbifold theories which study for example the fermionic and bosonic matter content of the resulting Yang-Mills theory, one naturally takes interest in Spin(6), or the full $\mathbb{Z}_2 \times \mathbb{Z}_2$ cover of SO(6) which admits spinor representations; for these we shall look in particular at the groups that have $\phi = 4$ in the Jordan notation, as will be indicated in the tables below.

3. The discrete finite subgroups of $\mathrm{SL}(4;\mathbb{C})$

We shall henceforth let Γ denote a finite subgroup of $\mathrm{SL}(4;\mathbb{C})$ unless otherwise stated.

3.1 Primitive subgroups

There are in all 30 types of primitive cases for Γ. First we define the constants $w = e^{2\pi i/3}$, $\beta = e^{2\pi i/7}$, $p = \beta + \beta^2 + \beta^4$, $q = \beta^3 + \beta^5 + \beta^6$, $s = \beta^2 + \beta^5$, $t = \beta^3 + \beta^4$, and $u = \beta + \beta^6$. Furthermore we shall adhere to some standard notation and denote the permutation and the alternating permutation group on n elements respectively as S_n and A_n. Moreover, in what follows we shall use the function $Lift$ to mean the lifting by (perhaps a subgroup) of the abelian center C according to the exact sequence $0 \to C \to \mathrm{SU}(4) \to \mathrm{SU}(4)/C \to 0$.

[3]See [4, 5] for a discussion on this point.

[4]For $n = 2$, this our familiar $\mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(3)$.

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We present the relevant matrix generators as we proceed:

$$F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w^2 \end{pmatrix} \; F_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & -1 \end{pmatrix} \; F_3 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$F_2' = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix} \; F_3' = \frac{1}{4} \begin{pmatrix} -1 & \sqrt{15} & 0 & 0 \\ \sqrt{15} & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} \; F_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta^4 & 0 \\ 0 & 0 & 0 & \beta^2 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad W = \frac{1}{i\sqrt{7}} \begin{pmatrix} p^2 & 1 & 1 & 1 \\ 1 & -q & -p & -p \\ 1 & -p & -q & -p \\ 1 & -p & -p & -q \end{pmatrix}$$

$$R = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & s & t & u \\ 2 & t & u & s \\ 2 & u & s & t \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w^2 \end{pmatrix} \quad D = \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$V = \frac{1}{i\sqrt{3}} \begin{pmatrix} i\sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & w & w^2 \\ 0 & 1 & w^2 & w \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We see that all these matrix generators are unitary except R.

3.1.1 Primitive simple groups

There are 6 groups of this most fundamental type:

Group	Order	Generators	Remarks
I$*$	60×4	F_1, F_2, F_3	$Lift(A_5)$
II$*$	60	F_1, F_2', F_3'	$\cong A_5$
III$*$	360×4	F_1, F_2, F_3	$Lift(A_6)$
IV$*$	$\frac{1}{2}7! \times 2$	S, T, W	$Lift(A_7)$
V	168×4	S, T, R	
VI$*$	$2^6 3^4 5 \times 2$	T, C, D, E, F	

3.1.2 Groups having simple normal primitive subgroups

There are 3 such groups, generated by simple primitives and the following 2 matrices:

$$F' = \frac{1+i}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad F'' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The groups are then:

Group	Order	Generators	Remarks
VII$*$	120×4	(I), F''	Lift(S_5)
VIII$*$	120×4	(II), F'	Lift(S_5)
IX$*$	720×4	(III), F''	Lift(S_6)

3.1.3 Groups having normal intransitive subgroups

There are seven types of Γ in this case and their fundamental representation matrices turn out to be Kronecker products of those of the exceptionals of SU(2). In other words, for M, the matrix representation of Γ, we have $M = A_1 \otimes_K A_2$ such that A_i are the 2×2 matrices representing $E_{6,7,8}$. Indeed we know that $E_6 = \langle S_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}\rangle$, $E_7 = \langle S_{\mathrm{SU}(2)}, U_{\mathrm{SU}(2)}\rangle$, $E_8 = \langle S_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}, V_{\mathrm{SU}(2)}\rangle$, where

$$S_{\mathrm{SU}(2)} = \frac{1}{2}\begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix} \qquad U_{\mathrm{SU}(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}.$$

$$V_{\mathrm{SU}(2)} = \begin{pmatrix} \frac{i}{2} & \frac{1-\sqrt{5}}{4} - i\frac{1+\sqrt{5}}{4} \\ -\frac{1-\sqrt{5}}{4} - i\frac{1+\sqrt{5}}{4} & -\frac{i}{2} \end{pmatrix}$$

We use, for the generators, the notation $\langle A_i\rangle \otimes \langle B_j\rangle$ to mean that Kronecker products are to be formed between all combinations of A_i with B_j. Moreover the group (XI), a normal subgroup of (XIV), is formed by tensoring the 2-by-2 matrices

$$x_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad x_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}, \quad x_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix},$$
$$x_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}, \quad x_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}, \quad x_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}.$$

The seven groups are:

Group	Order	Generators	Remarks
X$*$	144×2	$\langle S_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}\rangle \otimes \langle S_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}\rangle$	$\cong E_6 \otimes_K E_6$
XI$*$	288×2	$x_1 \otimes x_2, x_1 \otimes x_2^T, x_3 \otimes x_4, x_5 \otimes x_6$	$(\mathrm{X}) \lhd \Gamma \lhd (\mathrm{XIV})$
XII$*$	288×2	$\langle S_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}\rangle \otimes \langle S_{\mathrm{SU}(2)}, U_{\mathrm{SU}(2)}\rangle$	$\cong E_6 \otimes_K E_7$
XIII$*$	720×2	$\langle S_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}\rangle \otimes \langle S_{\mathrm{SU}(2)}, V_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}\rangle$	$\cong E_6 \otimes_K E_8$
XIV$*$	576×2	$\langle S_{\mathrm{SU}(2)}, U_{\mathrm{SU}(2)}\rangle \otimes \langle S_{\mathrm{SU}(2)}, U_{\mathrm{SU}(2)}\rangle$	$\cong E_7 \otimes_K E_7$
XV$*$	1440×2	$\langle S_{\mathrm{SU}(2)}, U_{\mathrm{SU}(2)}\rangle \otimes \langle S_{\mathrm{SU}(2)}, V_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}\rangle$	$\cong E_7 \otimes_K E_8$
XVI$*$	3600×2	$\langle S_{\mathrm{SU}(2)}, V_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}\rangle \otimes \langle S_{\mathrm{SU}(2)}, V_{\mathrm{SU}(2)}, U^2_{\mathrm{SU}(2)}\rangle$	$\cong E_8 \otimes_K E_8$

3.1.4 Groups having X-XVI as normal primitive subgroups

There are in all 5 of these, generated by the above, together with

$$T_1 = \frac{1+i}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

The group generated by (XIV) and T_2 is isomorphic to (XXI), generated by (XIV) and T_1 so we need not consider it. The groups are:

Group	Order	Generators
XVII$*$	576×4	$(\mathrm{XI}), T_1$
XVIII$*$	576×4	$(\mathrm{XI}), T_2$
XIX$*$	288×4	$(\mathrm{X}), T_1$
XX$*$	7200×4	$(\mathrm{XVI}), T_1$
XXI$*$	1152×4	$(\mathrm{XIV}), T_1$

3.1.5 Groups having normal imprimitive subgroups

Finally these following 9 groups of order divisible by 5 complete our list of the primitive Γ, for which we need the following generators:

$$A = \frac{1+i}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \frac{1+i}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$S' = \frac{1+i}{\sqrt{2}}\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T' = \frac{1+i}{2}\begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix} \quad R' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & -i \end{pmatrix}$$

Moreover these following groups contain the group K of order 16×2, generated by:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We tabulate the nine groups:

Group	Order	Generators
XXII$*$	$5 \times 16 \times 4$	$(\mathrm{K}), T'$
XXIII$*$	$10 \times 16 \times 4$	$(\mathrm{K}), T', R'^2$
XXIV$*$	$20 \times 16 \times 4$	$(\mathrm{K}), T, R$
XXV$*$	$60 \times 16 \times 4$	$(\mathrm{K}), T, S'B$
XXVI$*$	$60 \times 16 \times 4$	$(\mathrm{K}), T, BR'$
XXVII$*$	$120 \times 16 \times 4$	$(\mathrm{K}), T, A$
XXVIII$*$	$120 \times 16 \times 4$	$(\mathrm{K}), T, B$
XXIX$*$	$360 \times 16 \times 4$	$(\mathrm{K}), T, AB$
XXX$*$	$720 \times 16 \times 4$	$(\mathrm{K}), T, S$

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3.2 Intransitive subgroups

These cases are what could be constructed from the various combinations of the discrete subgroups of $\mathrm{SL}(2;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{C})$ according to the various possibilities of diagonal embeddings. Namely, they consist of those of the form $(1,1,1,1)$ which represents the various possible abelian groups with one-dimensional (cyclotomic) representation,[5] $(1,1,2)$, two abelians and an $\mathrm{SL}(2;\mathbb{C})$ subgroup, $(1,3)$, an abelian and an $\mathrm{SL}(3;\mathbb{C})$ subgroup, and $(2,2)$, two $\mathrm{SL}(2;\mathbb{C})$ subgroups as well as the various permutations thereupon. Since these embedded groups (as collineation groups of lower dimension) have been well discussed [5], we shall not delve too far into their account.

3.3 Imprimitive groups

The analogues of the dihedral groups (in both $\mathrm{SL}(2;\mathbb{C})$ and $\mathrm{SL}(3;\mathbb{C})$), which present themselves as infinite series, are to be found in these last cases of Γ. They are of two subtypes:

(a) Generated by the canonical abelian group of order n^3 for $n \in \mathbb{Z}^+$ whose elements are

$$\Delta = \left\{ \begin{pmatrix} \omega^i & 0 & 0 & 0 \\ 0 & \omega^j & 0 & 0 \\ 0 & 0 & \omega^k & 0 \\ 0 & 0 & 0 & \omega^{-i-j-k} \end{pmatrix} \right\} \quad \begin{matrix} \omega = e^{2\pi i/n} \\ i,j,k = 1,\ldots,n \end{matrix}$$

as well as respectively the four groups A_4, S_4, the Sylow-8 subgroup $Sy \subset S_4$ (or the ordinary dihedral group of 8 elements) and $\mathbb{Z}_2 \times \mathbb{Z}_2$;

(b) We define H and T'' (where again $i = 1,\ldots,n$) as:

$$H = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} \qquad T'' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^i & 0 & 0 & 0 \\ 0 & \omega^{-i} & 0 & 0 \end{pmatrix}$$

where the blocks of H are $\mathrm{SL}(2;\mathbb{C})$ subgroups.

We tabulate these last cases of Γ as follows:

Subtype	Group	Order	Generators
(a)	XXXI$*$	$12n^3$	$\langle \Delta, A_4 \rangle$
	XXXII$*$	$24n^3$	$\langle \Delta, S_4 \rangle$
	XXXIII$*$	$8n^3$	$\langle \Delta, Sy \rangle$
	XXXIII$*$	$4n^3$	$\langle \Delta, \mathbb{Z}_2 \times \mathbb{Z}_2 \rangle$
(b)	XXXIV$*$		$\langle H, T'' \rangle$

[5] These includes the $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_p$ groups recently of interest in brane cube constructions [17].

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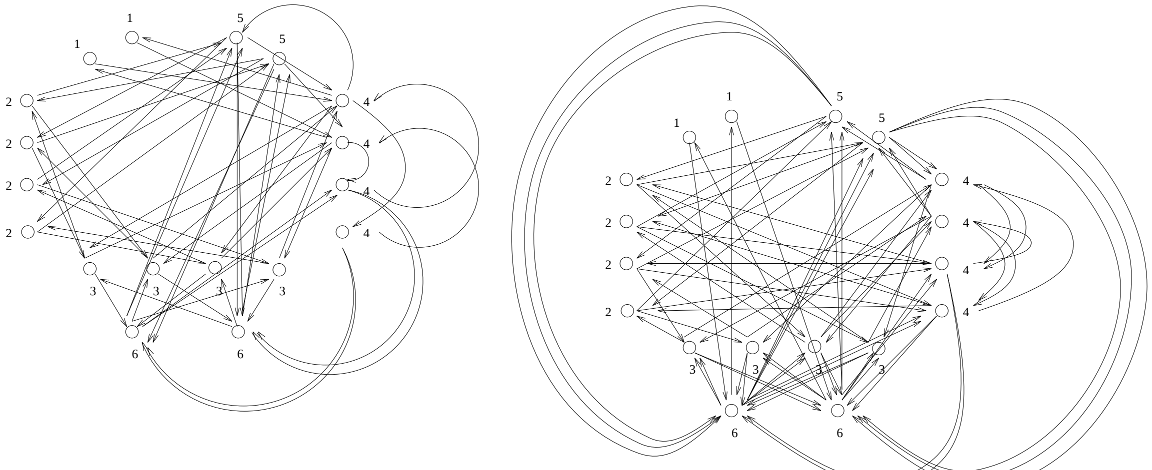

(a) Fermionic (b) Bosonic

Figure 1: The Quiver Diagram for Group (I), constructed for (a) the fermionic a^4_{ij} corresponding to the irreducible $\mathbf{4}_3$ and (b) the bosonic a^6_{ij} corresponding to the irreducible $\mathbf{6}_2$ (in the notation of [5]). We make this choice because we know that $\mathbf{4}_1 \otimes \mathbf{4}_3 = \mathbf{4}_3 \oplus \mathbf{6}_1 \oplus \mathbf{6}_2$ and that the two **6**'s are conjugates. The indices are the dimensions of the various irreducible representations, a generalisation of Dynkin labels.

4. Remarks

We have presented, in modern notation, the classification of the discrete subgroups of $\mathrm{SL}(4, \mathbb{C})$ and in particular, of SU(4). The matrix generators and orders of these groups have been tabulated, while bearing in mind how the latter fall into subcategories of transitivity and primitivity standard to discussions on collineation groups.

Furthermore, we have computed the character table for the 30 exceptional cases [18]; The interested reader may, at his or her convenience, find the character tables at `http://pierre.mit.edu/~yhe/su4.ct`. These tables will be crucial to quiver theories. As an example, we present in figure 1 the quiver for the irreducible **4** of the group (I) of order 60×4, which is the lift of the alternating permutation group on 5 elements.

Indeed such quiver diagrams may be constructed for all the groups using the character tables mentioned above. We note in passing that since $\Gamma \subset \mathrm{SU}(4)$ gives rise to an $\mathcal{N} = 0$ theory in 4 dimensions, supersymmetry will not come to our aid in relating the fermionic $a^{\mathbf{4}}_{ij}$ and the bosonic $a^{\mathbf{6}}_{ij}$ as was done in [5]. However we can analyse the problem with a slight modification and place a stack of M2 branes on the orbifold, (which in the Maldacena picture corresponds to orbifolds on the S^7 factor in $AdS_4 \times S^7$), and obtain an $\mathcal{N} = 2$ theory in 3 dimensions at least in the IR limit as we lift from type-IIA to M- theory [8, 11, 12]. This supersymmetry would help us to impose the constraining relation between the two matter matrices, and hence the two quiver diagrams. This would be an interesting check which we leave to future work.

We see therefore a host of prospective research in various areas, particularly in the context of string orbifold/gauge theories, WZW modular invariants, and singularity-resolutions in algebraic geometry. It is hoped that this monograph, together with its companion tables on the web, will provide a ready-reference to works in these directions.

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Ad Catharinae Sanctae Alexandriae et ad Majorem Dei Gloriam ...

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